Notes on the Second Eigenvalue of the Google Matrix

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February 1, 2008

Abstract

If A is an $n \times n$ matrix whose n eigenvalues are ordered in terms of decreasing modules, $|\lambda_1| \geq |\lambda_2| \geq \cdots |\lambda_n|$, it is often of interest to estimate $\frac{|\lambda_2|}{|\lambda_1|}$. If A is a row stochastic matrix (so $\lambda_1 = 1$), one can use an old formula of R. L. Dobrushin to give a useful, explicit formula for $|\lambda_2|$. The purpose of this note is to disseminate these known results more widely and to show how they imply, as a very special case, some recent theorems of Haveliwala and Kamvar about the second eigenvalue of the Google matrix.

If $A = (a_{ij})$ is an $n \times n$ real matrix, A has n (counting algebraic multiplicity) complex eigenvalues which can be listed in order of decreasing modules: $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$. We have $|\lambda_1| = \sup\{|\lambda| | \lambda$ is an eigenvalue of $A\}$ and $|\lambda_1|$ is called the spectral radius of $A, r(A) := |\lambda_1|$. In many problems it is of interest to estimate $\frac{|\lambda_2|}{|\lambda_1|} = \frac{|\lambda_2|}{r(A)}$. Indeed, an analogous problem is of great interest for bounded linear maps on Banach spaces: see [2], [3] and the references there.

Slightly more generally, suppose that V is an m-dimensional real vector space and $L : V \to V$ is a linear map. Again L has m possibly complex eigenvalues which can be written in order of decreasing modules: $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_m|$. If $\| \cdot \|$ denotes any norm on V (recall that all norms on a finite dimensional real vector space give the same topology), we can define

$$\|L\| = \sup\{\|Ly\| : y \in V, \|y\| \leq 1\}. \tag{1}$$

It is known that

$$\begin{aligned} r(L) = |\lambda_1| &= \sup\{|\lambda| : \lambda \text{ is an eigenvalue of } L\} \\ &= \lim_{k\to\infty} \|L^k\|^{\frac{1}{k}} = \inf_{k\geq 1} \|L^k\|^{\frac{1}{k}}, \end{aligned} \tag{2}$$

where L^k denotes the composition of L with itself k-times.

*Partially supported by NSF DMS-00-70829

1991 AMS Mathematics Subject Classification: Primary 15A18, 15A42, 15A48

We shall consider elements of $\mathbb{R}^n$, as usual, as column vectors. An $n \times n$ matrix B induces a linear map $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$ by $\Lambda(y) = By$. If V is a vector subspace of $\mathbb{R}^n$ and $By \in V$ for all $y \in V$, then B induces a linear map $L : V \to V$ by $L(y) = By$ for $y \in V$. If $\dim(V) = m$, then L has (counting algebraic multiplicity) precisely m eigenvalues, and these are the eigenvalues of B whose corresponding eigenvectors lie in the complexification of V.

Now suppose that $A = (a_{ij})$ is an $n \times n$ row stochastic matrix, so $a_{ij} \geq 0$ for all i, j and $\sum_{j=1}^{n} a_{ij} = 1$ for $1 \leq i \leq n$. Denote by x^t the transpose of a vector x and by B^t the transpose of a matrix B. If $e = (1, 1, \cdots, 1)^t$, then $Ae = e$, so $1 \in \sigma(A)$, where $\sigma(A)$, the spectrum of A, denotes the collection of eigenvalues of A. Recall that (in general) $\sigma(A) = \sigma(A^t)$, so $1 \in \sigma(A^t)$. It follows that (in general) $r(A) = r(A^t)$; and it is an elementary fact (the proof is sketched below) that $r(A) = 1$ for A row stochastic.

It will be convenient to use the L^1 norm $\|\cdot\|$, on $\mathbb{R}^n$, so for

$$\begin{aligned} y &= (y_1, y_2, \cdots, y_n)^t \in \mathbb{R}^n \\ \|y\|_1 : &= \sum_{i=1}^{n} |y_i|. \end{aligned} \tag{3}$$

Using the L^1 norm, we get a corresponding norm on $n \times n$ matrices $B = (b_{ij})$, since these matrices induce linear maps:

$$\|B\|_1 = \sup\{\|By\|_1 : \|y\|_1 \leq 1, y \in \mathbb{R}^n\}. \tag{4}$$

Indeed, using this norm when A is row stochastic, it is easy to see that $\|A^t\|_1 = 1$. Since $1 \in \sigma(A^t)$, we deduce, using eq. (2), that $r(A^t) = 1$ and hence $r(A) = 1$.

$$\text{If } x, y \in \mathbb{R}^n, \text{ let } \langle x, y \rangle = \sum_{i=1}^{n} x_i y_i.$$

Lemma 1 *Let A be an $n \times n$ row stochastic matrix and let $V = \{x = (x_1, x_2, \cdots, x_n)^t \in \mathbb{R}^n | \sum_1^n x_i = 0\}$. If $x \in V$, it follows that $A^t x \in V$.*

Proof: If $x \in V$, $\langle x, e \rangle = 0 = \langle x, Ae \rangle = \langle A^t x, e \rangle$, so $A^t x \in V$. ■

Henceforth, V will be as in Lemma 1.

If A is row stochastic, let $L : V \to V$ be the linear map induced by A^t. If $1 = \lambda_1 \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$ are the moduli of the eigenvalues of A^t, our previous remarks show that $\lambda_2, \lambda_3, \cdots, \lambda_n$ are the eigenvalues of L and eq. (2) implies that

$$|\lambda_2| := \text{ spectral radius of } L := r(L) = \lim_{k \to \infty} \|L^k\|_1^{\frac{1}{k}} = \inf_{k \geq 1} \|L^k\|_1^{\frac{1}{k}}. \tag{5}$$

By definition,

$$\begin{aligned} \|L^k\|_1 &= \sup\{\|(A^k)^t y\|_1 : y \in V, \|y\|_1 \leq 1\} \\ &= q(A^k). \end{aligned} \tag{6}$$

Note that A^k is a row stochastic matrix. If B is any row stochastic matrix, we follow (6) and define

$$\begin{aligned} q(B) &= \sup\{\|B^t y\|_1 : y \in V, \|y\|_1 \leq 1\}, \text{ where} \\ V &= \{y \in \mathbb{R}^n : \sum_1^n y_i = 0\}. \end{aligned} \tag{7}$$

The formula given by eqns (5)-(7) would be of limited usefulness without an explicit formula for $q(B)$. Fortunately, Dobrushin has given such a formula in Lemma 1, Section 3 of [1]; a slightly more general result is proved in Lemma 3.4 of [5].

Lemma 2 *(Dobrushin [1]). Let $B = (b_{ij})$ be an $n \times n$ row stochastic matrix and let $V = \{y \in \mathbb{R}^n | \sum_1^n y_i = 0\}$. If $q(B)$ is defined by (7), then*

$$q(B) = \left(\frac{1}{2}\right) \sup_{i,k} \left(\sum_{j=1}^n |b_{ij} - b_{kj}|\right) = 1 - \min_{i,k} \sum_{j=1}^n \min(b_{ij}, b_{kj}). \tag{8}$$

Combining the above observations we obtain a useful formula for $|\lambda_2|$ when A is row stochastic.

Theorem 1 *Let A be an $n \times n$ row stochastic matrix with eigenvalues $1 = \lambda_1, \lambda_2, \cdots, \lambda_n$, where $1 = |\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$ and eigenvalues are counted with algebraic multiplicity. (Recall that these eigenvalues are the same as the eigenvalues of A^t). Then we have*

$$|\lambda_2| = \lim_{k \to \infty} q(A^k)^{\frac{1}{k}} = \inf_{k \geq 1} q(A^k)^{\frac{1}{k}}, \tag{9}$$

where $q(B)$ is defined by eq. (8)

One can verify directly that for **any** $n \times n$ real matrix B, if we **define** $q(B) := (\frac{1}{2}) \sup \left(\sum_{j=1}^n |b_{ij} - b_{kj}|\right)$, then q is a seminorm, ie, $q(B + C) \leq q(B) + q(C)$ for any $n \times n$ matrices B and C and $q(\alpha B) = |\alpha| q(B)$ for any scalar α. Furthermore, if B and C are any $n \times n$ real matrices which have $e^t = (1, 1, \cdots, 1)^t$ as an eigenvector, then $q(BC) \leq q(B)q(C)$.

For the case that E is row stochastic of rank 1, the next result is proved in [4] by different methods.

Corollary 1 *Let P be an $n \times n$ row stochastic matrix and let E be an $n \times n$ row stochastic matrix. If $0 \leq c \leq 1$, let $A = cP + (1 - c)E$ and let λ_2 denote the second eigenvalue of A (where $1 = \lambda_1 \geq |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$). Then we have*

$$|\lambda_2| \leq cq(P) + (1 - c)q(E) \text{ and } |\lambda_2| \leq c \text{ if } E \text{ has rank } 1. \tag{10}$$

Proof: Theorem 1 implies that $|\lambda_2| \leq q(A) \leq cq(P) + (1 - c)q(E)$. If E has rank 1, then all rows of E are identical and $q(E) = 0$. Since $q(P) \leq 1$, this gives $q(A) \leq c$ when E has rank 1. ■

The following result is proved in [4] by a somewhat more involved argument.

Corollary 2 *Let P and E be as in Corollary 1 and assume E has rank 1. If $\{y \in \mathbb{R}^n | y = Py\}$ has dimension greater than one then $|\lambda_2| = c$, where λ_2 is the second eigenvalue of $cP + (1-c)E$. In fact, we have $\lambda_2 = c$.*

Proof: By assumption, there are linearly independent vectors v and w with $Pv = v$ and $Pw = w$. We also know there is a vector $z \in \mathbb{R}^n, \sum_{i=1}^n z_i = 1, z_i \geq 0$ for $1 \leq i \leq n$, such that $Ex = \langle x, z \rangle e$. If $\langle v, z \rangle = 0, (cP + (1-c)E)v = cPv = cv$, and if $\langle w, z \rangle = 0, (cP + (1-c)E)w = cw$. Thus assume that $\langle v, z \rangle \neq 0$ and $\langle w, z \rangle \neq 0$ and define $\xi = -\langle w, z \rangle v + \langle v, z \rangle w$. Note that $\langle \xi, z \rangle = 0$ and $\xi \neq 0$ because v and w are linearly independent. It follows that $(cP + (1-c)E)\xi = cP\xi = c\xi$. We conclude that c is an eigenvalue of $cP + (1-c)E$. Since we already know from Corollary 1 that $|\lambda_2| \leq c$ we conclude that $|\lambda_2| = c$ and that we can take $\lambda_2 = c$. ∎

Remark 1 The statement (see [4]) that "P has at least two irreducible closed subsets" is equivalent to the assertion that $\dim\{y \in \mathbb{R}^n | y = Py\} \geq 2$.

Remark 2 Suppose that A is row stochastic and $q(A) = \kappa < 1$. Let $\sum = \{x \in \mathbb{R}^n | \sum_1^n x_i = 1$ and $x_j \geq 0$ for $1 \leq j \leq n\}$. One easily checks that $A^t x \in \sum$ if $x \in \sum$, and our previous remarks show that $\|A^t x - A^t y\|_1 \leq \kappa \|x - y\|_1$ for all $x, y \in \sum$. By the contraction mapping theorem, for any $x \in \sum$, $\lim_{k \to \infty} (A^t)^k x = v$, where $A^t v = v$. Also, the rate of convergence can be estimated in terms of κ. The same assertions are true if $q(A^m) = \kappa_m < 1$ for some $m \geq 1$.

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